NECESSARY CONDITIONS FOR CONTINUOUS PARAMETER
STOCHASTIC OPTIMIZATION PROBLEMS

by

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1. Introduction.

This paper applies the abstract variational theory of Neustadt [1] to obtain a stochastic maximum principle. Since the papers of Kushner on the stochastic maximum principle [2], [3], a number of developments were reported in [illegible]odeau [4], Baum [5], Fleming [6], Sworder [7] - [8]. The versatile mathematical programming ideas were not used explicitly in [2] - [8], and, with relative ease, we are able to handle greater varieties of state space constraints then treated in the references. A discrete parameter analog of the discrete maximum principle of Halkin [9] and Holtzman [10] appears in Kushner [11]. Even in the deterministic case, the ability to handle general constraints with relative ease gives the programming approach a distinct advantage over more direct approaches.

It is premature to assert that the stochastic maximum principle will be useful in providing any deep understanding of stochastic control problems. Nevertheless, it seems likely that the implicit geometric framework (at least in the programming approach) will suggest some useful approximation or numerical procedures. The results may serve as a departure point for a perturbation analysis as in the formal work [12], and the nature and interpretation of the random multipliers may shed additional light on the physical interpretation of the derivatives (weak or strong) of the minimum cost function which appears in the dynamic programming formulation for a fully Markovian problem. These various points are under current investigation for both the present work and [11]. Even for an initially Markovian problem, dynamic programming is not always applicable when there are state space constraints, and the alternative programming formulation may be useful to shed light on the control problem. For a discussion, for an elementary stochastic control problem of the relationship between randomized controls and 'singular arcs' see [13].

The problem formulation and mathematical background is given in Section 2. A required result of Neustadt is stated in Section 3, the linearized equations are discussed in Section 4. Section 5 derives a certain convex cone. The maximum principle is stated in Section 6. The development in Sections 4-6 is for the open loop case and extensions are discussed in Section 7.

2. Problem Formulation and Mathematical Background.

Assumptions.

Let $z(t)$, $0 \leq t \leq T$ be a vector valued normalized Brownian motion on the probability triple $(\Omega, P(\cdot), \mathcal{B})$, where Ω is the sample space, and $P(\cdot)$ the measure on the σ-algebra $\mathcal{B}$ on Ω. For the vector $x = (x_1, \ldots, x_r)$ and matrix $\Phi = \{\varphi_{ij};\ i,j = 1, \ldots,\}$, define the Euclidean norms $|x|^2 = \sum_i |x_i|^2$, $|\Phi|^2 = \sum_{i,j} \varphi_{ij}^2$. The control system of concern is the stochastic differential equation (1) on[+] the time interval $[0,T]$

$$dx(t) = f(x(t),u(\omega,t),t)dt + \sigma(x(t),t)dz(t). \tag{1}$$

where z and x are $n+1$ vectors, and $x(t) = (x_o(t),\dots,x_n(t))$ where $x(0)$ is independent of $z(t) - z(s)$, $t \geq s \geq 0$, and also $E|x(0)|^2 < \infty$. Write $\sigma = [\sigma_o,\dots,\sigma_n]$, where σ_i is the i^{th} column of σ. We may write (1) as

$$dx(t) = f(x(t),u(\omega,t),t)dt + \sum_i \sigma_i(x(t),t)dz_i(t). \tag{2}$$

The cost is defined to be $\varphi_o(x(\cdot))$

$$\varphi_o(x(\cdot)) = Ex_o(T) + Eh(x_T), \tag{3}$$

and we impose the vector constraints

[+]Usually the ω argument of a random variable or function is omitted, and sometimes, when we write the differential form (1), the t argument will also be omitted. Also the prime ' on x' denotes transpose.

$$
\begin{aligned}
& r_o(x(\cdot)) \equiv E\tilde{r}_o(x(0)) = 0 \\
& q_o(x(\cdot)) \equiv E\tilde{q}_o(x(0),EX(0)) \leq 0 \\
(4) \qquad & q_i(x(\cdot)) = E\tilde{q}_i(x(t_i),EX(t_i)) \leq 0,\ i = 1,\ldots,k\ ,\ 0 < t_i < t_{i+1} < T \\
& r_T(x(\cdot)) = E\tilde{r}_T(x(T),EX(T)) = 0,\ q_T(x(\cdot)) = \\
& \qquad\qquad E\tilde{q}_T(x(T),EX(T)) \leq 0.
\end{aligned}
$$

It is assumed that (4) implies that $x_o(0) = 0$. As discussed below more general constraints can be treated. Let $\hat{u}(\omega,t)$ and $\hat{x}(\omega,t)$ denote the optimal control and the corresponding trajectory, resp.

Assume

(I-1) The family of admissible controls $\tilde{\mathscr{U}}$ is the collection of measurable random functions $u(\omega,t)$ with values in the set $\mathscr{U}_t$ at each $t \in [0,T]$. For each t, $u(\omega,t)$ is measurable with respect to the data σ-algebra $\mathscr{B}_t$, which is non-anticipative with respect to the $z(s)$ process. The initial condition $x(0)$ is measurable over $\mathscr{B}_o$ and $E|x(0)|^2 < \infty$. Let $\tilde{\mathscr{U}}$ contain at least one other point beside $\hat{u}$.

(I-2) The $f(x,u,t)$ and $\sigma_i(x,t)$ are Borel functions of their arguments, and are differentiable with respect to the components of x, and satisfy a growth condition of the type $|f(x,u,t)|^2 \leq K_o(1+|x|^2)$, $|\sigma_i(x,t)|^2 \leq K_o(1+|x|^2)$, uniformly in $u \in \mathscr{U}_t$ and t. The function $f(x,u,t)$ is continuous at each (x,u), uniformly in t. The Jacobians $f_x(x,u,t)$ and $\sigma_{i,x}(x,t)$ are uniformly bounded.

(I-3) For each $t \in (0,T]$ and $\mathscr{B}_t$ measurable and $\mathscr{U}_t$ valued variable, there is a $\delta(t) > 0$ so that for each $\delta < \delta(t)$ there is a random variable $\tilde{u}_{t-\delta}$ with the property that $\tilde{u}_{t-\delta}$ is measurable over each $\mathscr{B}_s$ and has values in each $\mathscr{U}_s$ where $s \in [t-\delta,t]$ and

$$(5) \qquad f(\hat{x}(t),u_t,t) - f(\hat{x}(t),\tilde{u}_{t-\delta},t) \to 0$$

in probability as $\delta \to 0$. Both $\tilde{u}_{t-\delta}$ and δ may depend on u_t and t.

Note. The condition of the last paragraph is included since we will use piecewise constant and non-anticipative perturbations to the optimal control. It asserts that the effect of any control u_t which is admissible at time t can be approximated by a control $u_{t-\delta}$ which is admissible at any point in the small interval $[t-\delta,t]$.

(I-4)

$$|q_i(x(t_i))| \leq K_o(1+E|x(t_i)|^2), \quad i = 0,1,\ldots,k,T$$
$$|r_i(x(t_i))| \leq K_o(1+E|x(t_i)|^2), \quad i = 0,T.$$

The $\tilde{q}_i(x,e)$ and $\tilde{r}_i(x,e)$ and $h(x)$ are Borel functions whose first derivatives with respect to each argument exist. Write $\hat{q}_{i,x}$, $\hat{q}_{i,e}$, $\hat{r}_{i,x}$, $\hat{r}_{i,e}$ for the Jacobians of $\tilde{q}_i(x,e)$ and $\tilde{r}_i(x,e)$ with respect to the first and second arguments (x and e) evaluated at $x = \hat{x}(t_i)$, $e = E\hat{x}(t_i)$. Write $\hat{h}_x$ for the gradient of $h(x)$ evaluated at $\hat{x}(T)$.

Suppose that the $q_i(x(\cdot))$ and $r_i(x(\cdot))$ (evaluated at $\hat{x}(t_i)$) and $Eh(x_T)$ (evaluated at $\hat{x}_T$) have Frechet derivatives Q_i, R_i, H, resp., of the form

$$Q_i v = E[\hat{q}_{i,x}\cdot v + \hat{q}_{i,e}\cdot Ev]$$

$$R_i v = E[\hat{r}_{i,x}\cdot v + \hat{r}_{i,e}\cdot Ev]$$

$$Hv = E\hat{h}_x'\cdot v$$

where v is an arbitrary $(n+1)$ vector with square integrable components. I.e., the Q_i, R_i and H are continuous linear (vector or scalar valued) functionals on the space of square integrable random variables, [since q_i and r_i depend only on $x(t_i)$, the Frechet derivative is appropriate] and, e.g.,

$$\frac{1}{\epsilon} E[\tilde{q}_i(\hat{x}(t_i) + \epsilon v, E\hat{x}(t_i) + E\epsilon v) - \tilde{q}_i(\hat{x}(t_i), E\hat{x}(t_i))] \to Q_i\cdot v$$

as $\epsilon \to 0$ uniformly for v in any bounded set $\{v: E|v|^2 \leq a < \infty\}$. The components of each of the vector valued linear functionals R_i are linearly independent.

(I-5) Let $q_i = (q_i^1, \ldots, q_i^{a_i})'$. For the inactive+ inequality constraints $q_i^j(x)$, suppose that there is some $\epsilon_i > 0$ so that

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+ q_i^j is active at $\hat{x}(\cdot)$ if $q_i^j(\hat{x}(\cdot)) = 0$. Otherwise, it is said to be inactive.

$$q_i^j(\hat{x}(t_i) + v) < 0$$

for $E|v|^2 < \epsilon_i$. For the active constraints $q_i^j(x)$ suppose that there is a square integrable v_i so that for each i and active q_i^j,

$$Q_i^j \cdot v_i < 0$$

where Q_i^j is the Frechet derivative of the constraint component $q_i^j(x)$ at $\hat{x}(t_i)$.

3. A Variational Result of Neustadt.

For future reference, we describe a variational result of Neustadt [1]. Let $\mathcal{T}$ denote a locally convex topological space which contains the set Q.

Definition. Let P^μ denote the set $\{\beta: \beta_i \geq 0, \sum_{i=1}^{\mu} \beta_i \leq 1\}$. Let K be a convex set in $\mathcal{T}$ which contains $\{0\}$ and some point other than $\{0\}$. Let $w_1, \dots, w_\mu$ be in K and let N be an arbitrary neighborhood of $\{0\}$. Let there exist an $\epsilon_o > 0$ (depending on $w_1, \dots, w_\mu$ and N) so that, for each ϵ in $(0, \epsilon_o]$, there is a continuous map $\zeta_\epsilon(\beta)$ from P^μ to $\mathcal{T}$ with the property

$$\zeta_\epsilon(\beta) \subset \{\epsilon(\sum_{i=1}^{\mu} \beta_i w_i + N)\} \cap Q.$$

Then K is a <u>first order convex approximation to</u> Q.

A Basic Optimization Problem.

Let $\mathcal{T}$ contain the set Q'. Find the element $\hat{w}$ in Q' which minimizes the real valued function $\varphi_o(w)$ subject to the real valued constraints $\varphi_i(w) = 0, i = 1,\ldots,\mu$, $\varphi_{-i}(w) \leq 0$, $i = 1,\ldots \beta$. We say that $\hat{w}$ is a <u>local solution</u> to the optimization problem (or, more loosely, the <u>optimal solution</u>) if, for some neighborhood N of $\{0\}$, $\varphi_o(w) \geq \varphi_o(\hat{w})$ for all w in $\hat{w} + N$ which satisfy the constraints. Let $\hat{w}$ denote the optimal solution. The constraints φ_{-i} for which $\hat{\varphi}_{-i} \equiv \varphi_{-i}(\hat{w}) = 0$ are called the <u>active constraints</u>. Define the set of indices $J = \{i: \varphi_{-i}(\hat{w}) = 0, i > 0\} \cup \{0\}$.

The Basic Necessary Condition for Optimality.

First we collect some assumptions

(II-1) The $\varphi_i(w)$, $i \geq 1$, are continuous at $\hat{w}$. There are continuous and linearly independent functionals $\ell_1,\ldots,\ell_\mu$ for which $[\varphi_i(\hat{w}+\epsilon w_n) - \varphi_i(\hat{w})]/\epsilon - \ell_i(w) \to 0$ as $\epsilon \to 0$ and for any bounded sequence $w_n \to w$ in $\mathcal{T}$.

(II-2) There is a neighborhood N of $\{0\}$ in $\mathcal{T}$ so that, for all <u>inactive</u> constraints, we still have $\varphi_{-i}(\hat{w}+w) < 0$ for $w \in N$.

(II-3) Let the active constraints and also φ_o be continuous at $\hat{w}$. For the active constraints, let

$$[\varphi_{-i}(\hat{w}+\epsilon w_n) - \varphi_{-i}(\hat{w})]/\epsilon \to c_i(w)$$

as $\epsilon \to 0$, for any bounded sequence $w_n \to w$ in $\mathcal{F}$, where $c_i(w)$ is a continuous and convex functional. There is some w and some $j \in J$ for which $c_j(w) > 0$. There is a w for which $c_j(w) < 0$ for all $j \in J$.

A case of particular importance is where the differentials $c_i(w)$ are linear functionals. Then the next to last sentence of (II-3) is implied by the last sentence of (II-3).

We now have a particular case of (Neustadt [1], Theorem 4.2). The local or optimal solution here is called a totally regular local solution in [1].

Theorem 1. Assume (II-1 - II-3). Let $\hat{w}$ be a local solution to the optimization problem. Then there exists $\alpha_1,\ldots,\alpha_\mu,\alpha_o,\alpha_{-1},\ldots,\alpha_{-\beta}$ not all zero with $\alpha_{-i} \le 0$ for $i \ge 0$, so that

$$\sum_{i=1}^{\mu} \alpha_i \ell_i(w) + \sum_{i \in J} \alpha_{-i} c_i(w) \le 0$$

for all w in $\bar{K}$, where K is a first order convex approximation to $Q' - \hat{w} \equiv Q$, and $\bar{K}$ is the closure of K in $\mathcal{F}$.

Remark. Let $\varphi_i(\cdot) \equiv 0$, $i > 0$. If there is a $w \in K$ for which $c_j(w) < 0$ for all active j, then $\alpha_o < 0$, and we can set $\alpha_o = -1$.

Identification with the Stochastic Control Problem.

For the problems of the sequel we define $\mathcal{T}$ to be the locally convex linear topological space of $(n+1)$ dimensional random functions v with values $v(\omega,t)$, where $v_n \to 0$ in $\mathcal{T}$ if and only if

$$E|v_n(\omega,t)|^2 \to 0$$

for+ each t in $[0,T]$. The set Q' is defined to be the set of solutions++ $x(\omega,t)$ to (1) for all admissible controls, and initial conditions satisfying $E|x(0)|^2 < \infty$, $x_o(0) = 0$, and $x(0)$ independent of $z(t) - z(s)$, $t \geq s \geq 0$. The constraints $\{q_i^j\}$ are identified with the $\{\varphi_{-\ell}, \ell > 0\}$ and the $\{r_i^j\}$ with the $\{\varphi_\ell, \ell > 0\}$. Also $\varphi_o = Ex_o(T) + Eh(x_T)$. $\hat{x}$ is the optimal element of Q' and $Q \equiv Q' - \{\hat{x}\}$. Conditions (I-1) - (I-8) imply (II-1) - (II-3).

With the framework of constraints (4), we can include constraints such as $E\int_0^{t_i} g_i(x(s))ds \leq d_i$ and can approximate constraints such as $P\{x(t) \in A\} \leq d_i$, where A has a smooth

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+It is easiest to work in the space of random functions $\mathcal{T}$, as it is described above. By (I-1), (I-2), we lose nothing by altering $\mathcal{T}$ so that $v_n \to 0$ if $E|v_n(\omega,t)|^p \to 0$ for any $p \geq 2$. In this case the quadratic estimates (I-4) on q_i and r_i can be replaced by $|q_i(x(t_i))| \leq K_o(1+E|x(t_i)|^p)$, etc. More general situations are obviously possible and, in particular, the Lipschitz and growth condition on the zeroeth component of $f(x,u,t)$ can be relaxed.

++$x, \hat{x}$ or v are elements of $\mathcal{T}$. Notation will be abused by also using either of $x(t)$ $x(\omega,t)$ for the element of $\mathcal{T}$, as well as for the values of the element.

boundary. More general inequality constraints than (4) can be included, once the appropriate linear or convex differentials c_i (see II-3) are calculated.

4. The Linearized Equations.

Consider the equations (6) and (7), where $0 \leq \tau \leq t \leq T$, τ is fixed and $\Phi(t,\tau)$ is an $(n+1) \times (n+1)$ matrix+

(6) $$dy(t) = \hat{f}_x \cdot y(t)dt + \sum_i dz_i(t)\hat{\sigma}_{i,x}y(t),$$

(7) $$d\Phi(t,\tau) = \hat{f}_x \Phi(t,\tau)dt + \sum_i dz_i(t)\hat{\sigma}_{i,x}\Phi(t,\tau),$$

where $\Phi(\tau,\tau) = I$, the identity, and $E|y(\tau)|^2 < \infty$ and $y(\tau)$ is independent of $z(t) - z(s)$, for all $t \geq s \geq \tau$. Both (6) and (7) have unique continuous (in t) solutions, with finite mean square values. A version of $\Phi(t,\tau)$ is measurable in (t,ω) for each τ. By uniqueness, for each $\tau \in [0,T]$, w.p.1.,

$$\Phi(t,\tau)y(\tau) = y(t)$$

and, for $t > \tau_1 > \tau$, w.p.1.,

$$\Phi(t,\tau_1)\Phi(\tau_1,\tau) = \Phi(t,\tau).$$

+ $\hat{f}_x$ denotes $f_x(\hat{x}(t),\hat{u}(t),t)$, etc.

Furthermore, $\Phi(t,\tau)$ is mean square continuous in τ, uniformly in $t \in [\tau,T]$. Indeed, we have w.p.1, that $\Phi(t,\tau+\epsilon)$ and $\Phi(t,\tau)$ are the solutions of (7) which start at time $\tau + \epsilon$ with initial values I and $\Phi(\tau+\epsilon,\tau)$, resp. By known estimates for solutions of stochastic differential equations, for real K_1,

$$E|\Phi(\tau+\epsilon,\tau) - I|^4 \leq K_1\epsilon^2$$

and, hence, for t in $[\tau+\epsilon,T]$

$$E|\Phi(t,\tau+\epsilon) - \Phi(t,\tau)|^4 \leq K_2\epsilon^2. \tag{8}$$

Equation (8) implies that there is a continuous version of $\Phi(T,\tau)$ (Proposition III.5.3 of [14]) (τ is the parameter). Finally, if $E|y(0)|^2 = O(\epsilon^2)$ (or $o(\epsilon^2)$), then $E|y(t)|^2 = O(\epsilon^2)$ (or $o(\epsilon^2)$). This last fact will be used frequently in Theorem 2.

5. The Convex Cone K.

We will require the following lemma.+

Lemma 1. Assume (I-1) - (I-2). Let the measurable function $\phi(\omega,t)$ be Lebesgue integrable on $[0,T]$ for almost all fixed ω. Then

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+The proof of Lemma 1 resulted from a discussion with W. Fleming.

$$F(\omega,t) = \int_{t_o}^{t} \phi(\omega,s)ds$$

is differentiable with respect to t on an (ω,t) set of full measure with derivative $\phi(\omega,t)$. Thus, there is a null set $T_1 \subset (0,T)$ so that, for each fixed $t \notin T_1$, $F(\omega,t)$ is differentiable with derivative $\phi(\omega,t)$, w.p.1. In particular, for $\phi(\omega,s) = f(\hat{x}(s),\hat{u}(s),s)$ and any scalars α_i, we have

$$\frac{1}{\epsilon}\int_{t-\epsilon\alpha_1}^{t+\epsilon\alpha_2} f(\hat{x}(s),\hat{u}(s),s)ds - (\alpha_2+\alpha_1)f(\hat{x}(t),\hat{u}(t),t) \to 0$$

w.p.1., for any t not in some null set T_1.

There is a null set $T_2 \subset (0,T)$ so that for $t \notin T_2$ and any random variable u,

$$\frac{1}{\epsilon}\int_{t-\epsilon\alpha_1}^{t+\epsilon\alpha_2} f(\hat{x}(s),u,s)ds - (\alpha_2+\alpha_1)f(\hat{x}(t),u,t) \to 0$$

w.p.1.

Proof. Define

$$F_r(\omega,t) = \frac{1}{r}\int_{t-\alpha_1 r}^{t+\alpha_2 r} \phi(\omega,s)ds,$$

where r is rational in $[0,1]$. There is a null ω set N_o so that, for $\omega \notin N_o$, $\phi(\omega,t)$ is Lebesgue integrable on $[0,T]$ and, hence for $\omega \notin N_o$

$$F_r(\omega,t) - (\alpha_1+\alpha_2)\phi(\omega,t) \to 0$$

for almost all t (the null t set depending on ω). Also $F_r(\omega,t)$ converges to $(\alpha_1+\alpha_2)\phi(\omega,t)$ on a measurable set $S \subset (\Omega-N_o) \times [0,T]$ as $r \to 0$. If $F_r(\omega,t)$ converges as $r \to 0$ through the rationals, it converges to the same limit as $r \to 0$ through any sequence.

The Lebesgue measure of the fixed ω sections of S (for $\omega \notin N_o$) is T. Hence by Fubini's theorem, the measurable set S has full measure. Thus, there is a null t set T_1 so that for $t \notin T_1$, $F_r(\omega,t) \to (\alpha_1+\alpha_o)\phi(\omega,t)$ w.p.1. The statements of the first paragraph of the lemma follow from this.

Let $g(v,t)$ denote a Borel function which is continuous at each v, uniformly in t. Let $g(v(t),t)$ be integrable on $[0,T]$ for any continuous $v(t)$. Then there is a null set T_2 so that, for $t \notin T_2$,

$$\frac{1}{\epsilon}\int_{t-\epsilon\alpha_1}^{t+\epsilon\alpha_2} g(v(s),s)ds - (\alpha_1+\alpha_2)g(v(t),t) \to 0$$

as $\epsilon \to 0$, for any continuous function $v(s)$. The second paragraph of the lemma follows from this by letting $f(\hat{x}(\omega,t),u(\omega),t) =$

$g(v(t,\omega),t)$, where $v(\omega,t) = (\hat{x}(\omega,t),u(\omega))$ and noting the continuity (in (x,u)) properties of $f(x,u,t)$ which were assumed in (I-2). Q.E.D.

The Convex Cone K.

Define the elements $\delta x_{s,u}$ of $\mathscr{T}$ (with values $\delta x_{s,u}(\omega,t)$, sometimes written as $\delta x_{s,u}(t)$), for $s \notin \hat{T} = T_1 \cup T_2$ (see Lemma 1 for the definition of T_i)

$$\begin{aligned} \delta x_{s,u}(t) &= 0, \qquad 0 \leq t < s \leq T \\ &= \Phi(t,s)[f(\hat{x}(s),u_s,s) - f(\hat{x}(s),\hat{u}(s),s)] \\ &\qquad\qquad T \geq t \geq s, \end{aligned}$$

where u_s is a $\mathscr{B}_s$ measurable random variable with values in $\mathscr{U}_s$. Define the set K as the convex cone (in $\mathscr{T}$) containing elements of the type

$$K = \{\delta x\colon \delta x = c_o \delta x_o + \sum_{i=1}^{d} c_i \delta x_{s_i,u_{s_i}}; c_i \text{ arbitrary, real, non-negative; } d \text{ arbitrary, finite}\},$$

where

$$\delta x_o(t) = \Phi(t,0)\delta x(0),$$

where $\delta x(0)$ is independent of $z(t) - z(s)$, all $t \geq s \geq 0$, and $E|\delta x(0)|^2 < \infty$. By Thecrem 1, K is a first order convex approximation to the set $Q' - \{\hat{x}\} \equiv Q$.

__Theorem__ 2. __Assume__ (I-1 - I-3). __Then__ K __is a first order convex approximation to__ $Q \equiv Q' - \{\hat{x}\}$.

__Proof.__ Define the set $\Lambda = \{\lambda = (\lambda_0,\ldots,\lambda_m): \lambda_i \geq 0, \sum_i \lambda_i \leq 1\}$. Let $\delta x^1,\ldots,\delta x^m$ denote any m elements of K. Then, writing u_j for u_{s_j},

$$\delta x^i = \sum_{j=1}^{q} \beta_{ij}\delta x_{s_j,u_j} + \sum_{j=1}^{q} \tilde{\beta}_{ij}\delta x_o^i$$

for some set of β_{ij}, $\tilde{\beta}_{ij}$, and sets $u_1,\ldots,u_q$, $S_1,\ldots,s_q$ and $\delta x_o^i(t) = \Phi(t,0)\delta x_o^i$. We assume that $s_i \leq s_{i+1}$. Any element in $\tilde{K}$, the convex hull of $(0,\delta x^1,\ldots,\delta x^m)$, corresponds to some $\lambda \in \Lambda$ and conversely, and has the form

$$\delta x_\lambda = \sum_{i=1}^{m} \lambda_i(\sum_{j=1}^{q}\beta_{ij}\delta x_{s_j,u_j}) + \sum_{i=1}^{m}\lambda_i(\sum_{j=1}^{q}\tilde{\beta}_{ij}\delta x_o^j) = \sum_{j=1}^{q}\delta t_j(\lambda)\delta_{x_{s_j,u_j}} + \sum_{j=1}^{q}\delta\tilde{t}_j(\lambda)\delta x_o^j$$

$$\delta t_j(\lambda) = \sum_{i=1}^{m}\beta_{ij}\lambda_i, \quad \delta\tilde{t}_j(\lambda) = \sum_{j=1}^{m}\tilde{\beta}_{ij}\lambda_i.$$

Note that $\epsilon\delta t_i(\lambda) = \delta t_i(\epsilon\lambda)$ for any scalar $\epsilon > 0$, and similarly for $\delta\tilde{t}_i(\lambda)$. Let $\tilde{O}(\epsilon^2)$ denote any random function v_ϵ for which $E|v_\epsilon(t)|^2 = O(\epsilon^2)$ for $t \in [0,T]$, and write $v_\epsilon = \tilde{o}(\epsilon^2)$ if $E|v_\epsilon(t)|^2 = o(\epsilon^2)$ for each $t \in [0,T]$. To prove the theorem we must show that there is an $\epsilon_o > 0$ so that, for each $\epsilon < \epsilon_o$, there is a continuous map $\zeta_\epsilon(\lambda)$ from Λ into $\mathscr{T}$ of the form

$$\zeta_\epsilon(\lambda) = \hat{x} + \epsilon\delta x_\lambda + \rho_{\epsilon,\lambda} \tag{9}$$

where $\rho_{\epsilon,\lambda} = \tilde{o}(\epsilon^2)$.

Next, a perturbed control and initial condition will be described. Suppose first that the s_i are distinct and $s_i \notin \hat{T} = T_1 \cup T_2$. Define $\tau = \sup_{i,\lambda\epsilon\Lambda} \delta t_i(\lambda)\cdot q$, and

$$I_i(\epsilon\lambda) = \{t\colon s_i - \epsilon\delta t_i(\lambda) < t \leq s_i\}.$$

There is an $\epsilon_o > 0$ so that for $\epsilon < \epsilon_o$ we have (i): the $I_i(\epsilon\lambda)$ are distinct, (ii): all $s_i - \epsilon\tau \geq 0$, (iii): $\epsilon\tau \leq \min\,[\delta(s_1),\ldots,\delta(s_q)]$, where $\delta(s)$ is defined by (I-3), and $\delta(s_i)$ corresponds to s_i and control u_i at s_i. Define the perturbed control $u_{\epsilon\lambda}(t)$

$$\begin{aligned} u_{\epsilon\lambda}(t) &= \hat{u}(t),\ t \notin \bigcup_i I_i(\epsilon\lambda) \\ &= \tilde{u}_{s_i-\epsilon\tau},\ t \in I_i(\epsilon\lambda), \end{aligned} \tag{10}$$

where $\tilde{u}_{s_i-\epsilon\tau}$ corresponds to u_{s_i} by (I-3), and as $\epsilon \to 0$, (5) of (I-3) holds.

If the s_i are not distinct, we follow the method for the deterministic problem [16] and define

$$\tau_i = \delta t_i(\lambda) + \cdots + \delta t_q(\lambda) \quad \text{if} \quad s_i = s_{i+1} \cdots = s_q$$

$$\tau_i = \delta t_i(\lambda) + \cdots + \delta t_r(\lambda) \quad \text{if} \quad s_i = s_{i+1} \cdots = s_r < s_{r+1},\ r < q.$$

$$\begin{aligned} I_i(\epsilon\lambda) &= \{t\colon s_i - \epsilon\tau_i < t \leq s_i - \epsilon\tau_i + \epsilon\delta t_i(\lambda)\} \\ &= \{t\colon s_i - \epsilon(\delta t_i(\lambda) + \cdots + \delta t_r(\lambda)) \\ &\qquad < t \leq s_i - \epsilon(\delta t_{i+1}(\lambda) + \cdots + \delta t_r(\lambda)). \end{aligned}$$

Then define $u_{\epsilon\lambda}(t)$ as in (10). Thus, if some s_i are identical, the intervals are shifted to the left.

By (I-1) and (I-3), the $u_{\epsilon\lambda}(t)$ is admissible. Let $x_{\epsilon\lambda} \in \mathcal{S}$ denote the solution of (1) for control $u_{\epsilon\lambda}(t)$ and initial condition

$$\text{(11)} \qquad \hat{x}(0) + \epsilon \sum_{j=1}^{q} \delta\tilde{t}_j(\lambda)\delta x_o^i(0) \equiv \hat{x}(0) + \epsilon\delta x_\lambda(0) \equiv x_{\epsilon\lambda}(0)$$

$$\epsilon\delta x_\lambda(0) = \delta x_{\epsilon\lambda}(0).$$

Define

$$(12) \qquad \zeta_\epsilon(\lambda) = x_{\epsilon\lambda}.$$

Fix ϵ in $(0,\epsilon_0)$. Let $\lambda(n) \to \lambda$ in Λ. Then $E|x_{\epsilon\lambda(n)}(0) - x_{\epsilon\lambda}(0)|^2 \to 0$, and the total length of the intervals on which $u_{\epsilon\lambda(n)}(t) \neq u_{\epsilon\lambda}(t)$ converges to zero. These facts imply that $E|x_{\epsilon\lambda(n)}(t) - x_{\epsilon\lambda}(t)|^2 \to 0$ for each t, which implies the continuity of $\zeta_\epsilon(\lambda)$ for each $\epsilon < \epsilon_0$. We need only prove the expansion (9), and this will be done in three parts.

1°. Let K_i denote real numbers.

$$(13a) \qquad d\hat{x}(t) = f(\hat{x}(t), \hat{u}(t), t)dt + \sum_j dz_j(t)\sigma_j(\hat{x}(t),t)$$

$$(13b) \qquad dx_{\epsilon\lambda}(t) = f(x_{\epsilon\lambda}(t), u_{\epsilon\lambda}(t), t)dt + \sum_j dz_j(t)\sigma_j(x_{\epsilon\lambda}(t),t)$$

$$(13c) \qquad dy_{\epsilon\lambda}(t) = \hat{f}_x y_{\epsilon\lambda}(t)dt + [f(\hat{x}(t), u_{\epsilon\lambda}(t), t) - f(\hat{x}(t), \hat{u}(t), t)] + \sum_j dz_j(t)\hat{\sigma}_{j,x} y_{\epsilon\lambda}(t)$$

$$y_{\epsilon\lambda}(0) = \delta x_{\epsilon\lambda}(0) = \epsilon\delta x_\lambda(0).$$

Using standard estimates it can be shown that, for some $K_1 < \infty$,

$$(14) \qquad \max_{\epsilon<\epsilon_0, \lambda\in\Lambda} E \max_{0\leq t\leq T} |x_{\epsilon\lambda}(t)|^2 \leq K_1.$$

Next, we show that

$$(15) \qquad E|\tilde{x}(t)|^2 \equiv E|\hat{x}(t) - x_{\epsilon\lambda}(t)|^2 = 0(\epsilon^2)$$

uniformly in t. Equation (15) holds for $t = 0$. Assume it holds for $t = t_o$, and that $u_{\epsilon\lambda}(t) = \hat{u}(t)$ for $t \in [t_o, t_1]$. We will show that (15) holds uniformly in $[t_o, t_1]$. Then, if (15) holds at $t = s_i - \epsilon\rho$, we show that it holds uniformly in $[s_i-\epsilon\rho, s_i]$, for any real ρ for which $s_i - \epsilon\rho \geq 0$. These two facts imply (15) as asserted. Let $\tilde{x}(t) = x_{\epsilon\lambda}(t) - \hat{x}(t)$. Then,

$$\tilde{x}(t) = \tilde{x}(t_o) + \int_{t_o}^{t} [f(x_{\epsilon\lambda}(s), \hat{u}(s), s) - f(\hat{x}(s), \hat{u}(s), s)]ds$$
$$+ \int_{t_o}^{t} [\sigma(x_{\epsilon\lambda}(s), s) - \sigma(\hat{x}(s), s)]dz(s),$$

where $E|\tilde{x}(t_o)|^2 = 0(\epsilon^2)$. By standard estimates using the Lipschitz condition,

$$E|\tilde{x}(t)|^2 \leq K_2|\tilde{x}(0)|^2 + K_2 \int_{t_o}^{t} E|\tilde{x}(s)|^2 ds$$

which implies (15) in $[t_o, t_1]$. Next, write

$$\tilde{x}(t) = \tilde{x}(s_i-\epsilon\rho) + \int_{s_i-\epsilon\rho}^{t} [f(x_{\epsilon\lambda}(s), u_{\epsilon\lambda}(s), s) - f(\hat{x}(s), \hat{u}(s), s)]ds$$
$$+ \int_{s_i-\epsilon\rho}^{t} [\sigma(x_{\epsilon\lambda}(s), s) - \sigma(\hat{x}(s), s)]dz(s).$$

Using the Lipschitz condition on σ, and Schwarz's inequality on the drift term, gives

$$E|\tilde{x}(t)|^2 \leq K_3 E|\tilde{x}(s_i-\epsilon\rho)|^2 + K_3 t \int_{s_i-\epsilon\rho}^{t} E[f(x_{\epsilon\lambda}(s),u_{\epsilon\lambda}(s),s) -$$

$$- f(\hat{x}(s),\hat{u}(s),s)]^2 ds + K_3 \int_{s_i-\epsilon\rho}^{t} |\tilde{x}(s)|^2 ds.$$

Using (14) and the growth condition $|f|^2 \leq K_o(1+|x|^2)$ in (I-2) gives

$$E|\tilde{x}(t)|^2 \leq K_3 E|\tilde{x}(s_i-\epsilon\rho)|^2 + K_4 t^2 + K_3 \int_{s_i-\epsilon\rho}^{t} E|\tilde{x}(s)|^2 ds$$

from which (15) follows in $[s_i-\epsilon\rho, s_i]$.

By reasoning close to the foregoing, it can be shown that

(16) $$E|y_{\epsilon\lambda}(t)|^2 = O(\epsilon^2)$$

uniformly in $t \in [0,T]$.

2^o. Next, it will be shown that

(17) $$E|x_{\epsilon\lambda}(t) - \hat{x}(t) - y_{\epsilon\lambda}(t)|^2 = o(\epsilon^2)$$

by the method used to show (15). Suppose $\hat{u}(t) = u_{\epsilon\lambda}(t)$ in $t \in [t_o,t_1]$ and (17) holds for $t = t_o$. Write $\tilde{y}(t) = x_{\epsilon\lambda}(t) - \hat{x}(t) - y_{\epsilon\lambda}(t)$. Then, for $t \in [t_o,t_1]$,

$$\tilde{y}(t) = \tilde{y}(t_o) + \int_{t_o}^{t} [f(x_{\epsilon\lambda}(s),\hat{u}(s),s) - f(\hat{x}(s),\hat{u}(s),s) - \hat{f}_x y(s)]ds$$

$$+ \int_{t_o}^{t} \sum_j dz_j(s)[\sigma_j(x_{\epsilon\lambda}(s),s) - \sigma_j(\hat{x}(s),s) - \hat{\sigma}_{j,x} y_{\epsilon\lambda}(s)]$$

$$(18) \qquad = \tilde{y}(t_o) + \int_{t_o}^{t} \hat{f}_x \tilde{y}(s)ds + \int_{t_o}^{t} \sum_j dz_j(s)\hat{\sigma}_{j,x}\tilde{y}(s) + e_1(t) + e_2(t),$$

where, for $\tilde{x}(s) = x(s) - \hat{x}(s)$,

$$e_1(t) = \int_{t_o}^{t} [f_x(\hat{x}(s) + \varphi(s)\tilde{x}(s),\hat{u}(s),s) - f_x(\hat{x}(s),\hat{u}(s),s)]\tilde{x}(s)ds$$

$$e_2(t) = \int_{t_o}^{t} \sum dz_j(s)[\sigma_{j,x}(\hat{x}(s) + \tilde{\varphi}(s)\tilde{x}(s),s) - \sigma_{j,x}(\hat{x}(s),s)]\tilde{x}(s)$$

where $\varphi(s)$ and $\tilde{\varphi}(s)$ are scalar valued random functions with values in $[0,1]$. By (15) and the continuity (in x) and boundedness properties of $f_x(x,u,s)$ and $\sigma_{i,x}(x,s)$,

$$E|e_i(t)|^2 = o(\epsilon^2)$$

uniformly in t. With this estimate (17) easily follows from (18) in $[t_o,t_1]$.

Next write $\delta t_i(\lambda) = \rho_i$ and let $E|\tilde{y}(s_i-\epsilon\tau_i)|^2 = o(\epsilon^2)$.

For $t \in [s_i-\epsilon\tau_i, s_i-\epsilon\tau_i+\epsilon\rho_i]$ write

$$(19)\qquad \tilde{y}(t) = \tilde{y}(s_i-\epsilon\tau_i) + \int_{s_i-\epsilon\tau_i}^{t} [f(x_{\epsilon\lambda}(s),u_{\epsilon\lambda}(s),s) - f(\hat{x}(s),\hat{u}(s),s) - \hat{f}_x y_{\epsilon\lambda}(s)]ds$$
$$- \int_{s_i-\epsilon\tau_i}^{t} [f(\hat{x}(s),u_{\epsilon\lambda}(s),s) - f(\hat{x}(s),\hat{u}(s),s)]ds$$
$$+ \int_{s_i-\epsilon\tau_i}^{t} \sum_j dz_j(s)[\sigma_i(x_{\epsilon\lambda}(s),s) - \sigma_i(\hat{x}(s),s) - \hat{\sigma}_{i,x}y_{\epsilon\lambda}(s)]$$

(19) can be written as

$$(20)\qquad \tilde{y}(t) = \tilde{y}(s_i-\epsilon\tau_i) + \int_{s_i-\epsilon\tau_i}^{t} [f(x_{\epsilon\lambda}(s),u_{\epsilon\lambda}(s),s) - f(\hat{x}(s),u_{\epsilon\lambda}(s),s)]$$
$$+ \int_{s_i-\epsilon\tau_i}^{t} \sum_j dz_j(s)[\sigma_j(x_{\epsilon\lambda}(s),s) - \sigma_j(\hat{x}(s),s)] + e_3(t)$$

$$e_3(t) = -[\int_{s_i-\epsilon\tau_i}^{t} \hat{f}_x y_{\epsilon\lambda}(s)ds + \int_{s_i-\epsilon\tau_i}^{t} \sum_j dz_j(s)\hat{\sigma}_{j,x}y_{\epsilon\lambda}(s)].$$

Using $E|y_{\epsilon\lambda}(s)|^2 = O(\epsilon^2)$ uniformly in s we get, for t in the desired interval,

$$E\left|\int_{s_i-\epsilon\tau_i}^{t} \sum_j dz_j(s)\hat{\sigma}_{j,x}y_{\epsilon\lambda}(s)\right|^2 \leq K_5 \int_{s_i-\epsilon\tau_i}^{t} |y_{\epsilon\lambda}(s)|^2 ds = o(\epsilon^2),$$

and similarly for the first term of $e_3(t)$. Using this and the estimates for the two integrals in (20),

$$E\left|\int_{s_i-\epsilon\tau_i}^{t} \sum_j dz_j(s)[\sigma_j(x_{\epsilon\lambda}(s),s) - \sigma_j(\hat{x}(s),s)]\right|^2 +$$

$$E\left|\int_{s_i-\epsilon\tau_i}^{t} [f(x_{\epsilon\lambda}(s),u_{\epsilon\lambda}(s),s) - f(\hat{x}(s),u_{\epsilon\lambda}(s),s)]ds\right|^2$$
$$\leq K_5 \int_{s_i-\epsilon\tau_i}^{t} E|\tilde{x}(s)|^2 ds,$$

and (15), gives (17) in $[s_i-\epsilon\tau_i,\ s_i-\epsilon\tau_i + \epsilon\rho_i]$.

3°. To complete the proof, it only remains to show that

$$E|y_{\epsilon\lambda}(t) - \delta x_{\epsilon\lambda}(t)|^2 = o(\epsilon^2) \tag{21}$$

(21) holds for $t = 0$, and indeed, (21) is zero for $t \in [0,s_1-\epsilon\tau_1]$. If (21) holds at t_o, then it is true in $[t_o,t_1]$ if $u_{\epsilon\lambda}(t) = \hat{u}(t)$ in $[t_o,t_1]$, since w.p.1

$$y_{\epsilon\lambda}(t_1) - \delta x_{\epsilon\lambda}(t_1) = \Phi(t_1,t_o)[y_{\epsilon\lambda}(t_o) - \delta x_{\epsilon\lambda}(t_o)]. \tag{22}$$

Next, for $t \in [s_i-\epsilon\tau_i,\ s_i-\epsilon\tau_i+\epsilon\rho_i]$, where $\rho_i = \delta t_i(\lambda)$,

$$y_{\epsilon\lambda}(t) = y_{\epsilon\lambda}(s_i-\epsilon\tau_i) + J_i^{\epsilon}(t) + \tilde{o}(\epsilon^2) \tag{23}$$

where

$$J_i^{\epsilon}(t) = \int_{s_i-\epsilon\tau_i}^{t} [f(\hat{x}(s),u_{\epsilon\lambda}(s),s) - f(\hat{x}(s),\hat{u}(s),s)]ds$$

Let $J_i^{\epsilon} \equiv J_i^{\epsilon}(s_i-\epsilon\tau_i+\epsilon\rho_i)$. If $y_{\epsilon\lambda}^{1}(t_o) = y_{\epsilon\lambda}(t_o) + \tilde{o}(\epsilon^2)$, and $u_{\epsilon\lambda}(t) = \hat{u}(t)$, $t \in [t_o,t_1]$, then, w.p.1.,

$$\tilde{y}_{\epsilon\lambda}^{1}(t_1) = \Phi(t_1,t_o)y_{\epsilon\lambda}(t_o) + \tilde{o}(\epsilon^2).$$

Furthermore, $E|J_i^{\epsilon}(t)|^2 = O(\epsilon^2)$ uniformly in t, and $\Phi(t-\epsilon\varphi_1,\ \tau-\epsilon\varphi_2) \to \Phi(t,\tau)$ in probability as $\epsilon \to 0$, for any constants φ_1,φ_2.

The last paragraph implies that w.p.1., for $t \notin \cup I_i^{o}(\epsilon\lambda)$, where $I_i^{o}(\epsilon\lambda)$ is the interior of $I_i(\epsilon\lambda)$,

$$y_{\epsilon\lambda}(t) = \sum_{t>s_i} \Phi(t,s_i)J_i^{\epsilon} + \Phi(t,0)\delta x_{\epsilon\lambda}(0) + \tilde{o}(\epsilon^2). \tag{24}$$

Define

$$J_i = \int_{s_i-\epsilon\tau_i}^{s_i-\epsilon\tau_i+\epsilon\rho_i} [f(\hat{x}(s),u_i,s) - f(\hat{x}(s),\hat{u}(s),s)]ds.$$

Then (I-3) implies that $E|J_i^\epsilon - J_i|^2 = o(\epsilon^2)$. Thus (24) is valid for J_i replacing J_i^ϵ. By Lemma 1, (letting $\alpha_2 = -\tau_i + \rho_i, \alpha_1 = \tau_i$)

$$\frac{1}{\epsilon\rho_i} J_i \to f(\hat{x}(s_i),u_i,s_i) - f(\hat{x}(s_i),\hat{u}(s_i),s_i)$$

w.p.1. as $\epsilon \to 0$.

Thus, for $t \notin \cup I_i^o(\epsilon\lambda)$,

$$\begin{aligned} y_{\epsilon\lambda}(t) &= \epsilon \sum_{t>s_i} \Phi(t,s_i)\delta t_i(\lambda)[f(\hat{x}(s_i),u_i,s_i) - f(\hat{x}(s_i),\hat{u}(s_i),s_i)] \\ &\qquad + \Phi(t,0)\delta x_{\epsilon\lambda}(0) + \tilde{o}(\epsilon^2) \qquad (25) \\ &= \delta x_{\epsilon\lambda}(t) + \tilde{o}(\epsilon^2). \end{aligned}$$

Since the sets $I_i^o(\epsilon\lambda)$ decrease to the empty set as $\epsilon \to 0$, (25) holds for all $t \in [0,T]$. Q.E.D.

6. The Maximum Principle.

Combining Theorems 1 and 2 we get Theorem 3. Define the column vector $P = (1,0,\ldots,0)'$. Theorem 3 reduces the Pontriagin maximum principle, if the noise is absent $(\sigma \equiv 0)$.

Theorem 3. Assume (I-1) - (I-5). There are continuous (in t) versions of $\Phi(T,t)$, $\Phi(t_i,t)$ (for $t \le T$ and $t \le t_i$, resp.). There is a scalar $\theta \le 0$, vectors $a_i \le 0$, $i = 0,1,\ldots,k,T$, (non-positive components a_i^j) where $a_i^j = 0$ if $q_i^j(\hat{x}) < 0$, and vectors

b_o, b_T, not all zero, so that for almost all $t \in [0,T]$ and all admissible $u(t)$, and $\mathscr{B}_o$ measurable $\delta x(0)$ satisfying $E|\delta x(0)|^2 < \infty$, (26) holds, w.p.l.

(26a)
$$\Big\{E\theta[P+h_x(\hat{x}(T))]'\Phi(T,t) + \sum_{t_i>t} E\, a_i'[\hat{q}_{i,x}+E\hat{q}_{i,e}]\Phi(t_i,t)$$
$$+ Eb_T'[\hat{r}_{T,x}+E\hat{r}_{T,e}]\Phi(T,t)\Big\}\cdot\Big\{f(\hat{x}(t), u(t),t) - f(\hat{x}(t),\hat{u}(t),t)\Big\} \leq 0$$

(26b)
$$\Big\{E\theta[P+h_x(\hat{x}_T)]'\Phi(T,0) + \sum_i a_i'E[\hat{q}_{i,x}+E\hat{q}_{i,e}]\Phi(t_i,0)$$
$$+ b_T'E[\hat{r}_{T,x}+E\hat{r}_{T,e}]\Phi(T,0) + b_o'E[\hat{r}_{o,x}+E\hat{r}_{o,e}]\Big\}\delta x(0) \leq 0$$

(26b) implies that the term in brackets is zero. Define the vector $p(T)$ by its transpose (27)

(27)
$$p'(T) = \theta[P+h_x(\hat{x}(T))]' + b_T'[\hat{r}_{T,x}+E\hat{r}_{T,e}] + a_T'[\hat{q}_{T,x}+E\hat{q}_{T,e}]$$

Define, where $t_o = 0$,

(28)
$$p'(t) = p'(T)\Phi(T,t), \qquad t_k \leq t \leq T$$
$$p'(t_i^-) = p'(t_i) + a_i'[\hat{q}_{i,x}+E\hat{q}_{i,e}], \qquad i = 1,\ldots,k$$
$$p'(t) = p'(t_i^-)\Phi(t_i,t), \quad t_{i-1} \leq t < t_i$$

Then (26) becomes

(29a) $$Ep'(t)[f(\hat{x}(t),u(t),t) - f(\hat{x}(t),\hat{u}(t),t)] \le 0$$

(29b) $$E[p'(0) + b_o'(\hat{r}_{o,x}+E\hat{r}_{c,e})]\delta x(0) = 0.$$

<u>Furthermore, w.p.1.</u>

(30a) $$E\{p'(t)[f(\hat{x}(t),u(t),t) - f(\hat{x}(t),u(t),t)]|\mathscr{B}_t\} \le 0$$

(30b) $$E\{[p'(0) + a_o'(\hat{r}_{o,x}+E\hat{r}_{o,e})]|\mathscr{B}_o\} = 0.$$

<u>Proof.</u> The proof of (26) follows from Theorem 1 using the identification of the Q_i, R_i with the c_i, ℓ_i in Theorem 1, and the fact that K is a first order convex approximation, by Theorem 2. Also the linear operator (acting on $\delta x(T)$), $E[P+h_x(\hat{x}_T)]'\delta x(T)$, is identified with c_o. Equation (29) follows from (26) upon using the substitution (27), (28). To prove (30a) suppose that (30a) is violated on a $\mathscr{B}_t$ measurable set B_t with $P(B_t) > 0$ Define $\bar{u}(t) = u(t)$ on B_t, $\bar{u}(t) = \hat{u}(t)$ on $\Omega - B_t$. Then (29a) is violated with the admissible $\bar{u}(t)$ replacing the $u(t)$ there. A similar proof yields (30b). Q.E.D.

7. <u>Extensions to Closed Loop Systems.</u>

Thus far the admissible $u(\omega,t)$ are defined to be measurable on the a priori fixed σ-algebra $\mathscr{B}_t$. If the admissible controls are assumed to depend explicitly on the state x - or its past values, i.e., $u(\omega,t) = u(x(t),t)$ or $u(\omega,t) = u(x_s, s \le t, t)$, then a very similar development can be carried out provided either the Lipschitz condition

$$(31) \qquad |u(x,t) - u(y,t)| \le K|x-y|$$

or the generalized Lipschitz condition, where x_t denotes the function with value $x_t(s) = x(t-s)$, $s \ge 0$,

$$|u(x_t,t) - u(y_t,t)| \le \int_0^M |x(t-s) - y(t-s)|dm(s)$$

for a bounded measure $m(\cdot)$ hold.+ Indeed, with the use of the perturbed controls and a convex cone K of the type used in Theorem 2, we obtain Theorem 3, with the exception that the $\hat{f}_x$ terms in the $y_{\epsilon\lambda}(t)$ and $\Phi(t,\tau)$ equations are replaced by $\hat{f}_x + \hat{f}_u \cdot \hat{u}_x$. In particular, let the data available to the control at time t be $g(x(t),t)$, where $g(x,t)$ is a Borel function satisfying (31) and $|g(x,t)|^2 \le K_o(1+|x|^2)$. Let the class of admissible controls U be the family of Borel functions $u(g,t)$ with values $u(g(x(t),t),t)$ and which satisfies (31), and which has values in $\mathscr{U}_t$ at time t. Assume the last two sentences of (I-1) and that (I-2) holds for each such u. The convex cone is composed of elements with values (for almost all s)

$$\Phi(t,0)\delta x(0) + \Phi(t,s)[f(\hat{x}(s),u(g(\hat{x}(s),s),s),s) - f(\hat{x}(s),\hat{u}(g(\hat{x}(s),s),s),s)].$$

It is supposed that there is a continuous function $\tilde{u}(g,t)$ satifying (31) with $\tilde{}(g,t) \in \mathscr{U}_t$ for all g and $|\tilde{u}(g,t)|^2 \le K_o(1+|g|^2)$

+For more detail on the more general stochastic differential delay system, see [15].

such that $u(g,s) = \tilde{u}(g,s)$. (This is not a significant restriction.)

Let $\tilde{u}_i(g,t)$ satisfy the conditions on $\tilde{u}$ above and reduce to $u(g,s_i)$ at $t = s_i$. In (10), let $u_{\epsilon\lambda}(\omega,t) = \tilde{u}_i(g(x(t),t),t)$ in $I_i(\epsilon\lambda)$. Then, under the additional conditions (I4-5), Theorem 3 holds with the conditioning on $\mathscr{B}$ replaced by conditioning on $g(\hat{x}(t),t)$. We have not given more details on the extensions, since attempts to extend the method to a more general class of controls whose members may be discontinuous in the state, have failed so far.

References

1. L. W. Neustadt, "An abstract variational theory with applications to a broad class of optimization problems", SIAM J. on Control, 4(1966), 505-527.

2. H. J. Kushner, "On the stochastic maximum principle: Fixed time of control", J. Math. Analysis and Applications, 11(1965), 78-92.

3. H. J. Kushner, "On the stochastic maximum principle with average constraints", J. Math. Analysis and Applications, 12(1965), 13-26.

4. F. Brodeau, "Contribution a'l'etude du controle optimal stochastique", Thesis, University of Grenoble, 1968.

5. R. F. Baum, "Optimal control systems with stochastic boundary conditions", Report 69-23, Dept. of Industrial Engrg., University of Michigan, Ann Arbor (1969).

6. W. Fleming, "Stochastic lagrange multipliers", Mathematical Theory of Control, ed. by A. V. Balakrishnan and L. W. Neustadt, Academic Press, New York, 1967.

7. D. D. Sworder, "On the control of stochastic systems", Int. J. Control, 6(1967), 179-188.

8. D. D. Sworder, "On the stochastic maximum principle", J. Math. Analysis and Applications, 24(1968), 627-640.

9. H. Halkin, "A maximum principle of the Pontriagin type for systems described by nonlinear difference equations", SIAM J. on Control, 4 (1966), 90-111.

10. J. M. Holtzman, "On the maximum principle for nonlinear discrete time systems", IEEE Trans. on Automatic Control, Vol. 4(1966), 528-547.

11. H. J. Kushner, "Necessary conditions for discrete parameter stochastic optimization problems", Brown University Center for Dynamical Systems Report 70-3, to appear in the Proceedings of the 6th Berkeley Symposium on Mathematical Statistics and Probability.

12. H. J. Kushner, "Near optimal control in the presence of small stochastic perturbations", J. Basic Engrg., 1965.

13. H. J. Kushner, and A. I. Kleinman, "Mathematical programming and the control of Markov chains", Brown University Center for Dynamical Systems Report 70-2, to appear in the Int. J. of Control.

14. J. Neveu, <u>Mathematical Foundations of the Calculus of Probability</u>, Holden Day, San Francisco, 1965.

15. W. Fleming and M. Nisio, "On the existence of optimal stochastic controls", J. Math. and Mech., <u>15</u>, 1966, 777-794.

16. M. D. Canon, C. D. Cullum, Jr., and E. Polak, <u>Theory of Optimal Control and Programming</u>, New York, McGraw-Hill, 1970.